Problem A.29

Consider the following hermitian matrix:

$$\mathsf{T} = \begin{pmatrix} 2 & i & 1 \\ -i & 2 & i \\ 1 & -i & 2 \end{pmatrix}.$$

(a) Calculate $\det(\mathsf{T})$ and $\operatorname{Tr}(\mathsf{T})$.

(b) Find the eigenvalues of T. Check that their sum and product are consistent with (a) in the sense of Equation A.93. Write down the diagonalized version of T.

(c) Find the eigenvectors of T. Within the degenerate sector, construct two linearly independent eigenvectors (it is this step that is always possible for a *hermitian* matrix, but not for an *arbitrary* matrix—contrast Problem A.19). Orthogonalize them, and check that both are orthogonal to the third. Normalize all three eigenvectors.

(d) Construct the unitary matrix S that diagonalizes T, and show explicitly that the similarity transformation using S reduces T to the appropriate diagonal form.

Solution

Take the hermitian conjugate of T.

$$\mathsf{T}^\dagger = \widetilde{\mathsf{T}}^* = \begin{pmatrix} 2 & -i & 1 \\ i & 2 & -i \\ 1 & i & 2 \end{pmatrix}^* = \begin{pmatrix} 2 & i & 1 \\ -i & 2 & i \\ 1 & -i & 2 \end{pmatrix}$$

Since $\mathsf{T}^\dagger = \mathsf{T}$, T is hermitian. As a result, the eigenvalues of T are real, the eigenvectors associated with distinct eigenvalues are orthogonal, and the matrix T is diagonalizable.
Solve the eigenvalue problem for T.

$$\mathsf{Ta} = \lambda \mathsf{a}$$

Bring $\lambda \mathsf{a}$ to the left side and factor a.

$$(\mathsf{T} - \lambda \mathsf{I})\mathsf{a} = \mathsf{0} \tag{1}$$

$\mathsf{a} \neq \mathsf{0}$, so the matrix in parentheses must be singular, that is,

$$\det(\mathsf{T} - \lambda \mathsf{I}) = 0$$

$$\begin{vmatrix} 2-\lambda & i & 1 \\ -i & 2-\lambda & i \\ 1 & -i & 2-\lambda \end{vmatrix} = 0$$

$$(2-\lambda)\begin{vmatrix} 2-\lambda & i \\ -i & 2-\lambda \end{vmatrix} - i \begin{vmatrix} -i & i \\ 1 & 2-\lambda \end{vmatrix} + \begin{vmatrix} -i & 2-\lambda \\ 1 & -i \end{vmatrix}$$

$$(2-\lambda)[(2-\lambda)(2-\lambda) - (-i)(i)] - i[(-i)(2-\lambda) - i] + [(-i)(-i) - (2-\lambda)] = 0$$

$$-9\lambda + 6\lambda^2 - \lambda^3 = 0.$$

Solve for λ.

$$-\lambda(\lambda-3)^2 = 0$$

$$\lambda = \{0, 3\}$$

Let $\lambda_0 = 0$ and $\lambda_+ = 3$. To find the eigenvectors corresponding to these eigenvalues, plug λ_0 and λ_+ back into equation (1).

$$(\mathsf{T} - \lambda_0 \mathsf{I})\mathsf{a}_0 = \mathsf{0} \qquad\qquad (\mathsf{T} - \lambda_+ \mathsf{I})\mathsf{a}_+ = \mathsf{0}$$

$$\begin{pmatrix} 2 & i & 1 \\ -i & 2 & i \\ 1 & -i & 2 \end{pmatrix}\begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \qquad \begin{pmatrix} -1 & i & 1 \\ -i & -1 & i \\ 1 & -i & -1 \end{pmatrix}\begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\left.\begin{aligned} 2a_1 + ia_2 + a_3 &= 0 \\ -ia_1 + 2a_2 + ia_3 &= 0 \\ a_1 - ia_2 + 2a_3 &= 0 \end{aligned}\right\} \qquad \left.\begin{aligned} -a_1 + ia_2 + a_3 &= 0 \\ -ia_1 - a_2 + ia_3 &= 0 \\ a_1 - ia_2 - a_3 &= 0 \end{aligned}\right\}$$

$$\left.\begin{aligned} a_3 &= -2a_1 - ia_2 \\ -ia_1 + 2a_2 + ia_3 &= 0 \\ a_1 - ia_2 + 2a_3 &= 0 \end{aligned}\right\} \qquad a_1 = ia_2 + a_3$$

$$a_1 - ia_2 + 2(-2a_1 - ia_2) = 0$$

$$-3a_1 - 3ia_2 = 0$$

$$a_1 = -ia_2$$
$$a_3 = -2(-ia_2) - ia_2 = ia_2$$

$$\mathsf{a}_0 = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} = \begin{pmatrix} -ia_2 \\ a_2 \\ ia_2 \end{pmatrix} \qquad \mathsf{a}_+ = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} = \begin{pmatrix} ia_2 + a_3 \\ a_2 \\ a_3 \end{pmatrix}$$

$$\mathsf{a}_0 = a_2\begin{pmatrix} -i \\ 1 \\ i \end{pmatrix} \qquad \mathsf{a}_+ = a_2\begin{pmatrix} i \\ 1 \\ 0 \end{pmatrix} + a_3\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$$

The three eigenvectors of T are then

$$\mathsf{a}_1 = A_1\begin{pmatrix} -i \\ 1 \\ i \end{pmatrix} \quad \text{and} \quad \mathsf{a}_2 = A_2\begin{pmatrix} i \\ 1 \\ 0 \end{pmatrix} \quad \text{and} \quad \mathsf{a}_3 = A_3\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix},$$

where the constants, A_1 and A_2 and A_3, are arbitrary due to the fact that the eigenvalue problem is homogeneous.

Notice that a_1 is orthogonal to both a_2 and a_3, but a_2 is not orthogonal to a_3.

$$\begin{aligned}\langle \mathsf{a}_1 \,|\, \mathsf{a}_2 \rangle &= \mathsf{a}_1^\dagger \mathsf{a}_2 \\ &= A_1^* \begin{pmatrix} i & 1 & -i \end{pmatrix} A_2 \begin{pmatrix} i \\ 1 \\ 0 \end{pmatrix} \\ &= A_1^* A_2 (0) \\ &= 0\end{aligned}$$

$$\begin{aligned}\langle \mathsf{a}_1 \,|\, \mathsf{a}_3 \rangle &= \mathsf{a}_1^\dagger \mathsf{a}_3 \\ &= A_1^* \begin{pmatrix} i & 1 & -i \end{pmatrix} A_3 \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \\ &= A_1^* A_3 (0) \\ &= 0\end{aligned}$$

$$\begin{aligned}\langle \mathsf{a}_2 \,|\, \mathsf{a}_3 \rangle &= \mathsf{a}_2^\dagger \mathsf{a}_3 \\ &= A_2^* \begin{pmatrix} -i & 1 & 0 \end{pmatrix} A_3 \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \\ &= A_2^* A_3 (-i) \\ &\neq 0\end{aligned}$$

In order to construct a new eigenvector that is orthogonal to the first two, use the Gram-Schmidt orthogonalization procedure. Subtract the component of a_3 along a_1 and the component of a_3 along a_2 off of a_3.

$$\begin{aligned}\mathsf{a}_4 &= \mathsf{a}_3 - \langle \mathsf{a}_1 \,|\, \mathsf{a}_3 \rangle \mathsf{a}_1 - \langle \mathsf{a}_2 \,|\, \mathsf{a}_3 \rangle \mathsf{a}_2 \\ &= \mathsf{a}_3 - (0)\mathsf{a}_1 - (-iA_2^* A_3)\mathsf{a}_2 \\ &= \mathsf{a}_3 + iA_2^* A_3 \mathsf{a}_2\end{aligned}$$

For the eigenvectors to be physically relevant, each of them must have a magnitude of one. That is, they must be normalized.

$$\begin{aligned}A_1^2 \left(|-i|^2 + |1|^2 + |i|^2 \right) &= 1 \\ A_1^2 (3) &= 1 \\ A_1 &= \pm \frac{1}{\sqrt{3}}\end{aligned}$$

$$\begin{aligned}A_2^2 \left(|i|^2 + |1|^2 + |0|^2 \right) &= 1 \\ A_2^2 (2) &= 1 \\ A_2 &= \pm \frac{1}{\sqrt{2}}\end{aligned}$$

$$\begin{aligned}A_3^2 \left(|1|^2 + |0|^2 + |1|^2 \right) &= 1 \\ A_3^2 (2) &= 1 \\ A_3 &= \pm \frac{1}{\sqrt{2}}\end{aligned}$$

Therefore, choosing the positive values for simplicity, the orthogonal normalized (orthonormal) eigenvectors of T are

$$\mathsf{a}_1 = \frac{1}{\sqrt{3}} \begin{pmatrix} -i \\ 1 \\ i \end{pmatrix} \quad \text{and} \quad \mathsf{a}_2 = \frac{1}{\sqrt{2}} \begin{pmatrix} i \\ 1 \\ 0 \end{pmatrix} \quad \text{and} \quad \frac{\mathsf{a}_4}{\|\mathsf{a}_4\|} = \frac{\mathsf{a}_3 + \frac{i}{2}\mathsf{a}_2}{\|\mathsf{a}_3 + \frac{i}{2}\mathsf{a}_2\|} = \frac{1}{\sqrt{\left|\frac{1}{2\sqrt{2}}\right|^2 + \left|\frac{i}{2\sqrt{2}}\right|^2 + \left|\frac{1}{\sqrt{2}}\right|^2}} \begin{pmatrix} \frac{1}{2\sqrt{2}} \\ \frac{i}{2\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix} = \frac{1}{\sqrt{6}} \begin{pmatrix} 1 \\ i \\ 2 \end{pmatrix},$$

where the first one is associated with $\lambda = 0$ and the latter two are associated with $\lambda = 3$. The negative values of A_1, A_2, and A_3 lead to acceptable answers as well.

In order to diagonalize T, let S^{-1} be the 3×3 matrix whose columns are the eigenvectors of T.

$$\mathsf{S}^{-1} = \begin{pmatrix} -\frac{i}{\sqrt{3}} & \frac{i}{\sqrt{2}} & \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} & \frac{i}{\sqrt{6}} \\ \frac{i}{\sqrt{3}} & 0 & \frac{2}{\sqrt{6}} \end{pmatrix}$$

Determine S, the similarity matrix, by finding the inverse of S^{-1}.

$$\left(\begin{array}{ccc|ccc} -\frac{i}{\sqrt{3}} & \frac{i}{\sqrt{2}} & \frac{1}{\sqrt{6}} & 1 & 0 & 0 \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} & \frac{i}{\sqrt{6}} & 0 & 1 & 0 \\ \frac{i}{\sqrt{3}} & 0 & \frac{2}{\sqrt{6}} & 0 & 0 & 1 \end{array}\right) \sim \left(\begin{array}{ccc|ccc} \frac{i}{\sqrt{3}} & 0 & \frac{2}{\sqrt{6}} & 0 & 0 & 1 \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} & \frac{i}{\sqrt{6}} & 0 & 1 & 0 \\ -\frac{i}{\sqrt{3}} & \frac{i}{\sqrt{2}} & \frac{1}{\sqrt{6}} & 1 & 0 & 0 \end{array}\right)$$

$$\sim \left(\begin{array}{ccc|ccc} \frac{i}{\sqrt{3}} & 0 & \frac{2}{\sqrt{6}} & 0 & 0 & 1 \\ 0 & \frac{1}{\sqrt{2}} & \frac{3i}{\sqrt{6}} & 0 & 1 & i \\ -\frac{i}{\sqrt{3}} & \frac{i}{\sqrt{2}} & \frac{1}{\sqrt{6}} & 1 & 0 & 0 \end{array}\right)$$

$$\sim \left(\begin{array}{ccc|ccc} \frac{i}{\sqrt{3}} & 0 & \frac{2}{\sqrt{6}} & 0 & 0 & 1 \\ 0 & \frac{1}{\sqrt{2}} & \frac{3i}{\sqrt{6}} & 0 & 1 & i \\ 0 & \frac{i}{\sqrt{2}} & \frac{3}{\sqrt{6}} & 1 & 0 & 1 \end{array}\right)$$

$$\sim \left(\begin{array}{ccc|ccc} \frac{i}{\sqrt{3}} & 0 & \frac{2}{\sqrt{6}} & 0 & 0 & 1 \\ 0 & \frac{1}{\sqrt{2}} & \frac{3i}{\sqrt{6}} & 0 & 1 & i \\ 0 & 0 & \frac{6}{\sqrt{6}} & 1 & -i & 2 \end{array}\right)$$

$$\sim \left(\begin{array}{ccc|ccc} \frac{i}{\sqrt{3}} & 0 & \frac{2}{\sqrt{6}} & 0 & 0 & 1 \\ 0 & \frac{1}{\sqrt{2}} & 0 & -\frac{i}{2} & \frac{1}{2} & 0 \\ 0 & 0 & \frac{6}{\sqrt{6}} & 1 & -i & 2 \end{array}\right)$$

$$\sim \left(\begin{array}{ccc|ccc} \frac{i}{\sqrt{3}} & 0 & 0 & -\frac{1}{3} & \frac{i}{3} & \frac{1}{3} \\ 0 & \frac{1}{\sqrt{2}} & 0 & -\frac{i}{2} & \frac{1}{2} & 0 \\ 0 & 0 & \frac{6}{\sqrt{6}} & 1 & -i & 2 \end{array}\right)$$

$$\sim \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{i}{\sqrt{3}} & \frac{1}{\sqrt{3}} & -\frac{i}{\sqrt{3}} \\ 0 & 1 & 0 & -\frac{i}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 1 & \frac{1}{\sqrt{6}} & -\frac{i}{\sqrt{6}} & \frac{2}{\sqrt{6}} \end{array}\right)$$

Consequently,

$$\mathsf{S} = \begin{pmatrix} \frac{i}{\sqrt{3}} & \frac{1}{\sqrt{3}} & -\frac{i}{\sqrt{3}} \\ -\frac{i}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{6}} & -\frac{i}{\sqrt{6}} & \frac{2}{\sqrt{6}} \end{pmatrix}.$$

Note that because the normalized eigenvectors were used for the columns of S^{-1}, this matrix is unitary, and S could have been found more conveniently by taking the hermitian conjugate of S^{-1}. Compute STS^{-1} and verify that T is diagonalizable.

$$\begin{aligned} \mathsf{STS}^{-1} &= \begin{pmatrix} \frac{i}{\sqrt{3}} & \frac{1}{\sqrt{3}} & -\frac{i}{\sqrt{3}} \\ -\frac{i}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{6}} & -\frac{i}{\sqrt{6}} & \frac{2}{\sqrt{6}} \end{pmatrix} \begin{pmatrix} 2 & i & 1 \\ -i & 2 & i \\ 1 & -i & 2 \end{pmatrix} \begin{pmatrix} -\frac{i}{\sqrt{3}} & \frac{i}{\sqrt{2}} & \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} & \frac{i}{\sqrt{6}} \\ \frac{i}{\sqrt{3}} & 0 & \frac{2}{\sqrt{6}} \end{pmatrix} \\ &= \begin{pmatrix} \frac{i}{\sqrt{3}} & \frac{1}{\sqrt{3}} & -\frac{i}{\sqrt{3}} \\ -\frac{i}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{6}} & -\frac{i}{\sqrt{6}} & \frac{2}{\sqrt{6}} \end{pmatrix} \begin{pmatrix} 0 & \frac{3i}{\sqrt{2}} & \frac{3}{\sqrt{6}} \\ 0 & \frac{3}{\sqrt{2}} & \frac{3i}{\sqrt{6}} \\ 0 & 0 & \sqrt{6} \end{pmatrix} \\ &= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{pmatrix} \end{aligned}$$

Calculate the determinant and trace of T.

$$\det(\mathsf{T}) = \begin{vmatrix} 2 & i & 1 \\ -i & 2 & i \\ 1 & -i & 2 \end{vmatrix} = 2\begin{vmatrix} 2 & i \\ -i & 2 \end{vmatrix} - i\begin{vmatrix} -i & i \\ 1 & 2 \end{vmatrix} + \begin{vmatrix} -i & 2 \\ 1 & -i \end{vmatrix} = 2(4+i^2) - i(-2i-i) + (i^2-2) = 0$$

$$\mathrm{Tr}(\mathsf{T}) = \mathrm{Tr}\begin{pmatrix} 2 & i & 1 \\ -i & 2 & i \\ 1 & -i & 2 \end{pmatrix} = 2+2+2 = 6$$

Calculate the determinant and trace of STS^{-1}.

$$\det(\mathsf{STS}^{-1}) = \begin{vmatrix} 0 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{vmatrix} = 0\begin{vmatrix} 3 & 0 \\ 0 & 3 \end{vmatrix} - 0\begin{vmatrix} 0 & 0 \\ 0 & 3 \end{vmatrix} + 0\begin{vmatrix} 0 & 3 \\ 0 & 0 \end{vmatrix} = 0$$

$$\mathrm{Tr}(\mathsf{STS}^{-1}) = \mathrm{Tr}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{pmatrix} = 0+3+3 = 6$$

Equation A.93 says that for a diagonalizable matrix T, the determinant is the product of the eigenvalues, and the trace is the sum of the eigenvalues.

$$\det(\mathsf{T}) = (0)(3)(3) = 0$$

$$\mathrm{Tr}(\mathsf{T}) = (0)+(3)+(3) = 6$$